Recursion Solutions

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Recursion Practice

1. Instead of 10 questions, simplify the problem with assuming a lesser number of questions. Let a_n represent the answer to the problem if there are n questions in the quiz. We want to find the answer when there are 10 questions, which means we want to find a_{10}.

   If $n = 1$, which means there is 1 question in this quiz, then there are 2 possible answer keys: T and F. For $n = 2$, there are 3: TT, TF, FT. Simply repeat the same for more values of n, and stop once it becomes too tedious to keep listing. Here is a chart showing the values for small values of n:

n	a_n
1	2
2	3
3	5
4	8
5	13

   Looking at the right side of the table, it's pretty clear what the pattern is: each number (besides a_1 and a_2) is the sum of the two previous numbers. In algebraic terms, $a_n = a_{n-1} + a_{n-2}$. Even if you don't understand why this pattern exists, you can still use it and be pretty sure you will get the right answer. To find a_{10}, use the pattern to continue the chart:

n	a_n
$\vdots$	$\vdots$
6	21
7	34
8	55
9	89
10	144

   So the total number of possible answer keys with 10 questions is $\boxed{144}$.

2. Let a_n represent the maximum number of pieces with n cuts. The answer we want to find is a_{10}. To find a pattern, we want to get as many small values as possible. The smallest value we can get isn't actually $n = 1$, but actually $n = 0$. The number of pieces with 0 cuts is 1 (the whole pizza). Here is a chart of the first few values:

n	a_n
0	1
1	2
2	4
3	7
4	11

The pattern is this: $a_1 - a_0 = 1$, $a_2 - a_1 = 2$, $a_3 - a_2 = 3$, $a_4 - a_3 = 4$. The difference between two consecutive a_n values goes up by 1. If we continue the chart using the pattern, you should get these values: $a_5 = 16$, $a_6 = 22$, $a_7 = 29$, $a_8 = 37$, $a_9 = 46$, $a_{10} = \boxed{56}$.

3. It's unreasonable to apply the recursive formula all the way to a_{100}, so we will make an explicit formula instead. Notice that for each a_n we have tried that it's simply the n-th triangular number plus 1. This allows us to write this explicit formula: $a_n = \frac{n(n+1)}{2} + 1$. This allows us to find a_{100} instantly: $a_{100} = \frac{100 \cdot 101}{2} + 1 = \boxed{5051}$.

4. Let a_n be the number of ways Jo can climb a flight of n stairs. Understand that $a_0 = 1$ because there is one way to climb 0 stairs: do nothing. In the future, remember that doing nothing may count as one way to do something. Here is a chart of n and a_n.

n	a_n
0	1
1	1
2	2
3	4
4	7
5	13

The pattern for values $n \geq 3$ is that a_n is the sum of the previous three terms. For the next values of a_n you should get $a_6 = 24$, $a_7 = 44$, $a_8 = \boxed{81}$.

5. This is technically called "telescoping," which is a technique not exactly like recursion, but similar in the fact that we will use small examples to create a pattern. Try the following small cases:

$$\frac{1}{1\cdot 2} = \frac{1}{2}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$

So it's pretty easy to see what the pattern is: if the last term is $\frac{1}{(n-1)n}$, then the sum of the whole series is $\frac{n-1}{n}$. Therefore, the answer to the original problem is $\boxed{\frac{99}{100}}$.

6. Simplify the question. Let n be the number of questions in the quiz, and let a_n be the number of possible answer keys. Something that may not be obvious is that $a_0 = 1$ (the number of possible keys with 0 questions). Having an empty set of answers is one way to make an answer key of 0 questions. For almost all recursion problems, $a_0 = 1$. Create the chart with n and a_n. I will also include a third column with a list of all the possible answer keys.

n	a_n	All possible answer keys
0	1	$\emptyset$ (a blank sequence of answers)
1	2	T, F
2	4	TT, TF, FT, FF
3	7	TTT, TTF, TFT, FTT, TFF, FTF, FFT
4	13	TTTT, TTTF, TTFT, TFTT, FTTT, TTFF, TFTF, TFFT, FTTF, FTFT, FFTT, FTFF, FFTF

The pattern for the a_n numbers is to add the previous three terms to get the next term. We can predict that $a_5 = 4 + 7 + 13 = 24$. Using this pattern, we can complete the sequence:

n	a_n
5	24
6	44
7	81
8	149
9	274
10	504

a_{10}, which means the answer to the problem if there are 10 questions, is $\boxed{504}$.

7. Using the recursive formula, we find $a_3 = \frac{3}{11}$, $a_4 = \frac{3}{15}$, and so on. It appears that $a_n = \frac{3}{4n-1}$, for all n. Setting $n = 2019$, we find $a_{2022} = \boxed{\dfrac{3}{8087}}$.

   To prove this formula, we use induction. We are given that $a_1 = 1$ and $a_2 = \frac{3}{7}$, which satisfy our formula. Now assume the formula holds true for all $n \le m$ for some positive integer m. By our assumption, $a_{m-1} = \frac{3}{4m-5}$ and $a_m = \frac{3}{4m-1}$. Using the recursive formula,

$$\begin{aligned} a_{m+1} &= \frac{a_{m-1} \cdot a_m}{2a_{m-1} - a_m} \\ &= \frac{\frac{3}{4m-5} \cdot \frac{3}{4m-1}}{2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}} \\ &= \frac{\left(\frac{3}{4m-5} \cdot \frac{3}{4m-1}\right)(4m-5)(4m-1)}{\left(2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}\right)(4m-5)(4m-1)} \\ &= \frac{9}{6(4m-1) - 3(4m-5)} \\ &= \frac{3}{4(m+1) - 1} \end{aligned}$$

   so our induction is complete.

8. Let n be the number of consecutive integers in the main set (e.g. $n = 3$ means {1,2,3}, and $n = 0$ means an empty set). Here is the chart of n and a_n.

n	a_n	All possible spacy subsets
0	1	$\emptyset$
1	2	$\emptyset$, {1}
2	3	$\emptyset$, {1}, {2}
3	4	$\emptyset$, {1}, {2}, {3}
4	6	$\emptyset$, {1}, {2}, {3}, {4}, {1,4}
5	9	$\emptyset$, {1}, {2}, {3}, {4}, {5}, {1,4}, {1,5}, {2,5}
6	13	$\emptyset$, {1}, {2}, {3}, {4}, {5}, {6}, {1,4}, {1,5}, {1,6}, {2,5}, {2,6}, {3,6}

   The pattern is that each term in a_n for $n \ge 3$ is the sum of the previous term and the term three before. $a_n = a_{n-1} + a_{n-3}$. So for example $a_7 = a_6 + a_4 = 13 + 6 = 19$. If you wish, you may check the number of spacy sets for $n = 7$, and you will get 19, confirming the pattern. Continue the table using this pattern.

n	a_n
7	19
8	28
9	41
10	60
11	88
12	129

a_{12} is the total number of spacy sets of 12 consecutive integers, which is the original problem. The answer is then $\boxed{\text{(E)}129}$.